FLUID FLOW BETWEEN POROUS PLANES

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On the basis of the solution of the equations of motion for the case of steady-state laminar flow of a viscous fluid between two circular or rectangular porous planes in the presence of a lateral filtration flow, we derived the theoretical relationships making it possible to determine the degree of filtration nonuniformity.

Let us examine the steady-state plane-parallel laminar flow of an incompressible viscous fluid between two parallel planes separated through a distance 2h in centimeters. One or both planes are permeable. The pressure outside the slot is constant, while the pressure inside the slot diminishes in the flow direction x. Filtration takes place within the slot or outside of the slot, under the influence of the pressure difference. Let us denote the excess pressure by p, so that $p > 0$ corresponds to filtration from the slot, while $p < 0$ corresponds to filtration within the slot. The system of the equations of motion and continuity for this case has the form

$$\left.\begin{aligned} &\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, && \text{(a)} \\ &\frac{\partial p}{\partial y} = 0, && \text{(b)} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. && \text{(c)} \end{aligned}\right\} \qquad (1)$$

In the case of impermeable walls we have the familiar solution

$$-\frac{dp}{dx} = q\,\frac{3\mu}{2h^3}. \qquad (2)$$

If the permeability of the walls is not great, the change in velocity along the flow - caused by the filtration - will be slow, the terms with second-order derivatives with respect to x will be small, and the equations of motion (1) will be retained, with the boundary conditions, however, being different.

We will assume that the local rate of filtration is proportional to the pressure with a constant permeability factor m. We will position the coordinate origin at the entry to the filtration segment on the slot axis. The boundary conditions for the velocity will then be

$$\begin{aligned} &x \geqslant 0, && y = \pm h, && u = 0; && \text{(a)} \\ &x \geqslant 0, && y = +h, && v = 0; && \text{(b)} \qquad (3) \\ &x \geqslant 0, && y = -h, && v = -mp. && \text{(c)} \end{aligned}$$

Having integrated (1a) for conditions (3a), we obtain

$$u = \frac{1}{2\mu}\,\frac{dp}{dx}\,(y^2 - h^2). \qquad (4)$$

Having substituted (4) into continuity equation (1c), after integration for condition (3c) we obtain the lateral velocity

$$v = \frac{1}{\mu}\,\frac{d^2p}{dx^2}\left(\frac{h^3}{3} + \frac{h^2 y}{2} - \frac{y^3}{6}\right) - mp. \qquad (5)$$

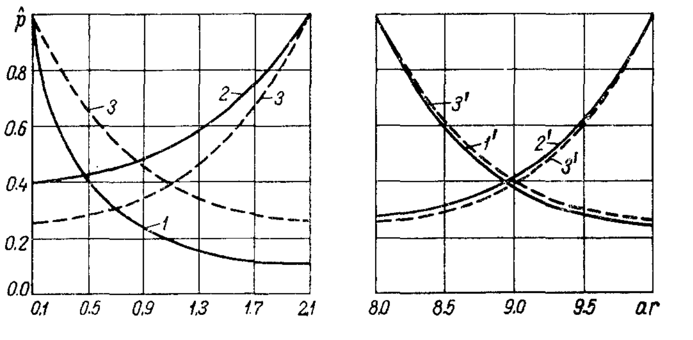
Condition (3b) for the determination of p yields the equation

$$\frac{d^2p}{dx^2} - \frac{3\mu m}{2h^3}\,p = 0, \qquad (6)$$

whose solution is

$$p = C_1 \operatorname{sh}(ax) + C_2 \operatorname{ch}(ax), \qquad (7)$$

where $a = (3\mu m/2h^3)^{0.5}$.



Pressure distribution in a liquid calculated from Eq. (36): 1 and 1') Flow from center; 2 and 2') flow toward center; 3 and 3') flow between rectangular planes (Eq. (18)).

If both planes are permeable, the entire solution remains in effect, but $a = (3\mu m/h^3)^{0.5}$.

The solution remains valid for $p > 0$ and for $p < 0$.

The formulation of the corresponding boundary conditions for p makes it possible to solve all of the problems associated with flow of this kind.

Let the initial excess pressure in the slot be given, as well as the flow rate in the supply circuit:

$$x = 0, \quad -h < y < h \quad \begin{cases} p = p_1, \\ q = q_1, \quad \text{or} \quad -\dfrac{dp}{dx} = q_1 \dfrac{3\mu}{2h^3}. \end{cases} \tag{8}$$

Condition (8) assumes that when $x < 0$ the flow takes place in a flat slot with impermeable walls, while when $x = 0$ the velocity profile is formed in accordance with (4).

Having determined the constants in (7) from condition (8), we obtain

$$p = p_1 \operatorname{ch}(ax) - \frac{aq_1}{m} \operatorname{sh}(ax). \tag{9}$$

If when $x = l$ the pressure p vanishes, l is a small length of the segment in which filtration from the slot is possible. Let us find this length. It follows from (9) that

$$\operatorname{th}(al) = \frac{mp_1}{aq_1} \tag{10}$$

or

$$l = \frac{1}{2a} \ln \frac{3\mu q_1 + 2ap_1h^3}{3\mu q_1 - 2ap_1h^3}, \tag{11}$$

and we can therefore regard as physically possible only those values of p_1 and q_1 which satisfy the inequality

$$p_1 < q_1 (3\mu/2h^3m)^{0.5}. \tag{12}$$

From (9) and (10) we also obtain

$$p = p_1 \frac{\operatorname{sh} a(l - x)}{\operatorname{sh} al}. \tag{13}$$

The total quantity of fluid drawn along path x through the porous plane is

$$q_f = \int_0^x v\,dx = -m \int_0^x p\,dx. \tag{14}$$

The remaining portion of the fluid $q_1 - q_f = q_{tr}$ represents the transit flow. Having substituted the value from (13) into (14) and after having performed the calculations, we find that

$$\operatorname{tr}_{\tau p} = q_1 \frac{\operatorname{ch} a(l - x)}{\operatorname{ch} al}. \tag{15}$$

It follows from (15) that the total filtration of the entire incoming fluid cannot be achieved in a segment of finite length. When $x = l$ the transit flow equals

$$q_{tr} = \frac{q_1}{\operatorname{ch} al}. \tag{16}$$

When $x > l$ and $p < 0$, the direction of filtration also changes

If we require total filtration on the segment $x = l_2$ the value of p is determined from (7) at the closed end of this segment for the boundary conditions:

$$\begin{aligned} &x = 0, \qquad p = p_1, \\ &x = l_2, \qquad q = 0, \quad \text{or} \quad \frac{dp}{dx} = 0. \end{aligned} \tag{17}$$

Then, for the pressure p we obtain

$$p = p_1 [\operatorname{ch} ax - \operatorname{th}(al_2) \operatorname{sh} ax] \tag{18}$$

and

$$\hat{p}_2 = \frac{p_2}{p_1} = \frac{1}{\operatorname{ch} al_2}. \tag{19}$$

Let us now examine the case of the radial flow of a fluid between permeable disks. The equations of motion and continuity are

$$\begin{aligned} &\frac{\partial p}{\partial r} = \mu \frac{\partial^2 v_r}{\partial z^2}, && \text{(a)} \\ &\frac{\partial p}{\partial z} = 0, \quad \frac{\partial p}{\partial \varphi} = 0, && \text{(b)} \\ &\frac{1}{r} \frac{\partial (rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0. && \text{(c)} \end{aligned} \tag{20}$$

The boundary conditions for the velocities are given by

$$\begin{aligned} &z = \pm h, \qquad v_r = 0, && \text{(a)} \\ &z = +h, \qquad v_z = 0, && \text{(b)} \\ &z = -h, \qquad v_z = -mp, && \text{(c)} \end{aligned} \tag{21}$$

then

$$v_r = \frac{1}{2\mu} \frac{dp}{dr} (z^2 - h^2),$$

$$v_z = -\left(\frac{d^2p}{dr^2} + \frac{1}{r} \frac{dp}{dr} \right) \times \left(\frac{z^3}{6\mu} - \frac{h^2 z}{2\mu} + \frac{h^3}{3\mu} \right).$$

To find the pressure from condition (21c) we obtain

$$\frac{d^2p}{dr^2} + \frac{1}{r} \frac{dp}{dr} - a^2 p = 0, \tag{22}$$

where $a^2 = 3\mu m/2h^3$.

The solution of Eq. (22) has the form

$$p = C_1 I_0(ar) + C_2 K_0(ar). \tag{23}$$

Let us examine certain special cases. We will assume everywhere in the following that $r_2 > r_1$.

1. Let the pressure p_1 and the flow rate q_1 along the radius r_1 for flow from the center to the periphery be given. In analogy with (8) we have

$$r = r_1, \quad p = p_1,$$

$$\frac{Q}{2\pi r} = q = q_1, \text{ or } -\frac{dp}{dr} = q\frac{3\mu}{2h^3} = q_1\frac{a^2}{m}. \quad (24)$$

From these conditions, having determined the constants in (23), we obtain

$$p = p_1(ar_1)\left\{\left[K_1(ar_1) - \frac{aq_1}{mp_1}K_0(ar_1)\right]I_0(ar) + \right.$$

$$\left. + \left[I_1(ar_1) + \frac{aq_1}{mp_1}I_0(ar_1)\right]K_0(ar)\right\}. \quad (25)$$

If with $r = r_*$ the pressure p vanishes, r_* is the greatest radius to which filtration from the slot is possible. This radius is determined from the condition that the expression in (25) contained within the braces is equal to zero when $r = r_*$. Hence we obtain

$$\frac{K_0(ar_*)}{I_0(ar_*)} = \frac{\frac{aq_1}{mp_1}K_0(ar_1) - K_1(ar_1)}{\frac{aq_1}{mp_1}I_0(ar_1) + I_1(ar_1)}. \quad (26)$$

We now calculate the total volume of the fluid which is filtered in the segment from r_1 to r_*:

$$Q_f = \int_{r_1}^{r_*} 2\pi rmp\,dr. \quad (27)$$

To calculate Q_f, instead of integrating (25), it is more convenient to determine the arbitrary constants in (23) from the conditions

$$r = r_1 \quad p = p_1,$$

$$r = r_* \quad p = 0, \quad (28)$$

which yield

$$p = p_1\frac{K_0(ar_*)I_0(ar) - I_0(ar_*)K_0(ar)}{I_0(ar_1)K_0(ar_*) - I_0(ar_*)K_0(ar_1)}, \quad (29)$$

and after the calculations we obtain

$$\frac{Q_f}{Q_1} = 1 - \frac{1}{ar_1[I_1(ar_1)K_0(ar_*) + I_0(ar_*)K_1(ar_1)]}. \quad (30)$$

The relationship between the incoming and filtered fluid volumes, or the magnitude of the transit flow $Q_{tr} = Q_1 - Q_f$, is a function of the quantities in the denominator in the right-hand member of (30). We will demonstrate that for any ar_1 it is greater than unity when $r_* > r_1$, i.e., that

$$ar_1[I_1(ar_1)K_0(ar_*) + I_0(ar_*)K_1(ar_1)] > 1$$

$$(r_* > r_1). \quad (31)$$

Inequality (31) is equivalent to the following:

$$\frac{I_1(ar_1)K_0(ar_*) + I_0(ar_*)K_1(ar_1)}{I_1(ar_1)K_0(ar_1) + I_0(ar_1)K_1(ar_1)} > 1$$

$$(r_1 < r < r_*). \quad (32)$$

Since $I_1(ar_1) \neq 0$ and $K_1(ar_1) \neq 0$, it is possible by means of term-by-term division to present (32) in the form

$$Y = \left[\frac{K_0(ar_*)}{K_1(ar_1)} + \frac{I_0(ar_*)}{I_1(ar_1)}\right] \Big/$$

$$\Big/ \left[\frac{K_0(ar_1)}{K_1(ar_1)} + \frac{I_0(ar_1)}{I_1(ar_1)}\right] > 1 \qquad (r_* > r_1). \quad (33)$$

When $r_* = r_1$, $Y \equiv 1$. Therefore, to prove inequality (33) it is sufficient that we establish that when $r_* > r_1$ the left-hand part of (33) increases with increasing r_*, or in other words, it is sufficient that w establish that $\frac{\partial Y}{\partial(ar_*)}$ is positive when $r_* > r_1$.

We find the derivative

$$\frac{\partial Y}{\partial(ar_*)} = \left[-\frac{K_1(ar_*)}{K_1(ar_1)} + \frac{I_1(ar_*)}{I_1(ar_1)}\right] \Big/$$

$$\Big/ \left[\frac{K_0(ar_1)}{K_1(ar_1)} + \frac{I_0(ar_1)}{I_1(ar_1)}\right]. \quad (34)$$

Since for all $r_* > r_1$, $K_1(ar_*) < K_1(ar_1)$, and $I_1(ar_*) > I_1(ar_1)$, the derivative is always positive. Hence it follows that we always have the condition $Q_f < Q_1$, i.e., total filtration is impossible on a radius of finite magnitude. Analogously, we can demonstrate that this statement is also true for flow toward the center.

2. If the slot is covered at radius r_2, i.e., if the disks are of limited dimension, and if the entire supplied volume of fluid is filtered, the boundary conditions for (22) are written in the form

$$r = r_1, \qquad -\frac{dp}{dr} = q_1\frac{a^2}{m},$$

$$r = r_2, \qquad \frac{dp}{dr} = 0, \quad (35)$$

and the pressure is expressed as

$$p = \frac{aq_1}{m}\frac{K_1(ar_2)I_0(ar) + I_1(ar_2)K_0(ar)}{I_1(ar_2)K_1(ar_1) - I_1(ar_1)K_1(ar_2)}. \quad (36)$$

Since the rate of filtration is proportional to the pressure, from (36) we derive the ratio of the local filtration flow rate at the radii r_1 and r_2, which is

$$\frac{Q_1}{Q_2} = ar_2[K_1(ar_2)I_0(ar_1) + I_1(ar_2)K_0(ar_1)]. \quad (37)$$

Expression (37) makes it possible to determine the degree of nonuniformity in filtration at the boundaries of a porous ring bounded by radii r_1 and r_2. The figure shows a number of curves for various values of ar_1 and ar_2 for identical initial flow rates Q_1.

As we can see from the shape of the curves, the nature of the pressure distribution differs substantially for cases of flow from the center and to the center. In the latter case the distribution is more uniform. With increasing distance from the center, the distribution of local filtration flows tends toward solution (18). As shown in the figure, the difference between solutions (36) and (18) does not exceed 5% when ar_2 = 10 and ar_1 = 8.

NOTATION

p is the pressure, dyne/cm^2; u and v are the velocity components in the direction of coordinate axes, cm/sec; q is the flow rate per unit length of the feed contour, cm^2/sec; μ is the fluid viscosity, g/cm sec; 2h is the width of the slot gap, cm; m is the permeability factor, cm^2 sec/g; r is the radius, cm; I_0 and K_0 are Bessel functions of imaginary argument with zero subscript; I_1 and K_1 are Bessel functions of imaginary argument with subscript 1; Q_f is the total volume of filtrated liquid, cm^3/sec; Q_{tr} is the transient volume of liquid through slot, cm^3/sec.

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